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THE UNIQUENESS OF THE INSTANTANEOUS FREQUENCY BASED ON INTRINSIC MODE FUNCTION

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It has been claimed that any expression of $a(t)\cos\theta(t)$ with $a(t)$ as the instantaneous amplitude and $\cos\theta(t)$ as the carrier varying along with the phase $\theta(t)$ could not be uniquely defined. However, based on the fact that $a(t)\cos\theta(t)$ with all its variational forms have the same numerical value at any given time, we propose the existence of a unique true intrinsic amplitude function $a_i(t)$ and phase function $\theta_i(t)$ that $a_i(t)\cos\theta_i(t)$ satisfying the envelope–carrier relationship is the only expression making physical sense. A constructive method is also presented to find such amplitude-phase pair uniquely from any Intrinsic Mode Function (IMF). As a result, we can treat any IMF in the form of $a_i(t)\cos\theta_i(t)$ as the unique defined amplitude-phase pair, from which the instantaneous frequency (IF) can also be determined.

Keywords: Empirical mode decomposition; intrinsic mode function; envelope; carrier; phase function; instantaneous frequency; AM/FM.

1. Introduction

The claim has been made by Huang *et al.* [2009] that a true time-frequency representation of any signal should be the Hilbert spectrum based on the instantaneous frequency (IF). To define the IF, the signal would have to be decomposed into a collection of Intrinsic Mode Functions (IMFs) through the Empirical Mode Decomposition (EMD) method or many of its variations for the first step [Huang *et al.* (1998); Wu and Huang (2009); Hou *et al.* (2009); Hou and Shi (2011, 2013); Rehman and Mandic (2010)]. Each IMF is expressed as:

$$c(t) = a(t)\cos\theta(t), \tag{1}$$

where $a(t)$ is the instantaneous amplitude, and $\cos\theta(t)$ is the associated carrier varying along with the phase function $\theta(t)$. Then, the IF is defined as:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(t). \tag{2}$$

Ever since the introduction of the notion of the IF, there are contentious discussions on both the mathematical and physical meaning of the term, in which a major one is the uniqueness of the expression on the right hand side (RHS) of Eq. (1) [Flandrin (1999)]. Taking the nonconstant function $b(t)$ restricted by $0 < |b(t)| < \min\{1, \min|a(t)|\}$ as an example, Eq. (1) equals to

$$c(t) = a(t)\cos\theta(t) = \frac{a(t)}{b(t)}[b(t)\cos\theta(t)]. \tag{3}$$

Setting

$$A(t) = \frac{a(t)}{b(t)}, \quad \Theta(t) = \cos^{-1}[b(t)\cos\theta(t)], \tag{4}$$

Eq. (3) is abbreviated as

$$c(t) = A(t)\cos\Theta(t), \tag{5}$$

where $A(t)\cos\Theta(t)$ represents one class of the variational forms of $a(t)\cos\theta(t)$. Applying the definition of IF, one will find

$$\omega(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \neq \frac{\mathrm{d}\Theta(t)}{\mathrm{d}t} = \frac{-1}{\sqrt{1-[b(t)\cos\theta(t)]^2}}\frac{\mathrm{d}[b(t)\cos\theta(t)]}{\mathrm{d}t}, \tag{6}$$

which suggests the nonuniqueness of the IF and therefore the nonsense of the term on the RHS of Eq. (1). This led Shekel [Shekel (1953)] to suggest that the term IF should be "banished forever from the dictionary of the communication engineer". It should be pointed out that even though the function could be expressed in infinite many different ways, the functional values, however, are uniquely determined and remain invariant. This seems to suggest that the different forms might be formal rather than substantial. One classic way to resolve this problem is proposed by Gabor [Gabor (1946)], who introduced an analytical form as the complex expression for the data:

$$z(t) = c(t) + \frac{\mathrm{i}}{\pi}PV\int_{-\infty}^{+\infty}\frac{c(t)}{t-\tau}\mathrm{d}\tau = c(t) + \mathcal{H}(c(t)), \tag{7}$$

with $\mathcal{H}(c(t))$ indicating the Hilbert transform of $c(t)$. This would give a unique imaginary part as an analytic extension of the real data. With this imaginary part defined, the amplitude is then given as

$$a(t) = [c(t) + (\mathcal{H}(c(t)))^2]^{\frac{1}{2}}, \tag{8}$$

through which the associated carrier could also be determined.

Unfortunately, there still exist some problems of using the Hilbert transform. First, the Hilbert transform requires the information of the data for all time due to the integration range covering the whole real axis. Therefore, the frequency defined through the Hilbert transform might not be local precisely. Additionally, the analytic pair is not the exact quadrature (defined here as shifting the phase by $\pi/2$). Generally speaking, even with the imaginary part given, the amplitude and the phase function in this way would be distorted. Furthermore, the theorems established by Bedrosian [Bedrosian (1963)] and Nuttall [Nuttall (1966)] had also severely limited the range of application of the Hilbert transform to the simplest phase function only. We will return to this point later.

Facing these difficulties, Huang *et al.* [2009] proposed the direct quadrature, yet the problem of nonuniqueness was not addressed explicitly. In a closer examination, all the confusions raised here, including the uniqueness problem stated in Eqs. (3)–(5), could be tackled easily with some clarifying definitions of the amplitude-carrier pair function. In the following sections, we will first establish the uniqueness rigorously through a series of definitions and constructive algorithms. Next, we will offer some examples to demonstrate subtlety of the differences between the traditional and the present approach. A short discussion and conclusion will then follow. An appendix is finally added to discuss the algorithm of constructing the true envelope for any oscillatory function.

2. Uniqueness of $c(t) = a(t)\cos\theta(t)$

The statement on the nonuniqueness of the RHS of Eq. (1) in the above is certainly valid. Then, how to define the IF from an expression like $a(t)\cos\theta(t)$? Is there a test or criterion for us to determine the amplitude function $a(t)$ and the associated carrier $\cos\theta(t)$ for a legitimate IF making physical sense? How can we ensure the uniqueness of such expression? The answer to all these questions are affirmative.

To begin with, the myriad forms of $A(t)\cos\Theta(t)$ from $a(t)\cos\theta(t)$ all have one invariant property: they have the same numerical value at any given time. We should use this invariant property to find a pair of unique functions, $a(t)$ and $\cos\theta(t)$ defined as the envelope and carrier. Mathematically, the definition of an envelope in geometry is given for a family of curves in the plane: an envelope is a curve tangent to each member of the family at some point. This envelope is defined as the amplitude. The procedure is as follows:

(1) Dive the data $c(t)$ in the IMF form into sections at local minima. This creates a family of wave cells, each of which starts from the minimum, then goes through the maximum and finally returns to the minimum.

(2) Construct an envelope e_v to this family of wave cells. Since the value of $c(t)$ is invariant, its envelope is also invariant (with respect to the $\mathbb{C}^2$-curve used) and hence unique. This envelope is defined as the true and intrinsic amplitude:

$$a_i(t) = e_v. \tag{9}$$

The associated carrier is defined as:

$$\frac{c(t)}{e_v} = \frac{a(t)\cos\theta(t)}{a_i(t)} = \cos\theta_i(t). \tag{10}$$

Considering the fact

$$\left|\frac{c(t)}{e_v}\right| \le 1$$

for all time, thus the carrier function would have values between ± 1, and the phase function can be expressed as an inverse cosine function:

$$\theta_i(t) = \cos^{-1}\left[\frac{a(t)\cos\theta(t)}{a_i(t)}\right]. \tag{11}$$

Therefore, if the function is separated into an amplitude and the associated carrier through the envelope based on the numerical value of $c(t)$, the separation would be unique. Correspondingly, the IF can also be determined as the time derivative of the phase function $\theta_i(t)$.

Any modification of the amplitude given by Eq. (5) would make the amplitude cease to be the envelop to the curve defined by $c(t)$, and hence it could not be the true intrinsic amplitude. Such a modified amplitude is designed as pseudo-amplitude. This condition can be demonstrated as follows. According to Eqs. (3) and (5), $A(t) = a(t)/b(t)$ would have values always greater than unity including all the points co-located at the same time as the maxima of the carrier; while,

$$|\cos\Theta(t)| = |b(t)\cos\theta(t)| < 1, \tag{12}$$

thus, the amplitude function $A(t)$ would not even touch the carrier function $\cos\Theta(t)$, and can never be a true envelope. Such a pair of amplitude and carrier are designed as pseudo-amplitude and pseudo-carrier function. With these definitions, we can state that the IF from IMF is uniquely defined by the true intrinsic amplitude-carrier pair.

The amplitude and carrier functions have also been viewed as the amplitude and frequency modulation (AM and FM) terms in engineering applications. Based on the above discussions, the only legitimate decomposition of the data into AM and FM terms is the one with true intrinsic amplitude as the AM part. The FM part is such a carrier that all its extrema equal to either $+1$ for all the maxima or -1 for all the minima. When extend this real valued data function to complex valued function using the direct quadrature, the imaginary part is defined as a shift of the phase angle of the real part by $\pi/2$.

In summary, among all the pseudo amplitude and carrier pairs, the true amplitude-carrier pair is the one that satisfies the following conditions: (1) the amplitude is the true envelope to the carrier function; (2) the carrier, with the imaginary part defined by the quadrature, will have a unit circle in the phase plane. Therefore, the following criteria should be able to separate the true and pseudo amplitude-carrier pair easily:

(1) Plot the function $c(t)$, whose value should be invariant.
(2) Define the amplitude function as the envelope to $c(t)$.
(3) Define the intrinsic phase as the inverse cosine function given by Eq. (11). The extrema of the intrinsic carrier function should be $+1$, otherwise, it is the pseudo one. In other words, the phase function for the carrier should be a unit circle.
(4) In case the expression is consisted of the pseudo amplitude and carrier, it still should have the same functional values. The true intrinsic amplitude and carrier could be determined with the normalization scheme outlined in Eqs. (9)–(11) easily.

3. Some Examples of True and Pseudo Amplitude and Carrier Functions

Now, let us consider some examples of true and pseudo amplitude and carrier pairs. Without loss of generality, we will illustrate the difference between the true and pseudo envelope through the simplest case. We first take a constant unit-amplitude cosine wave with constant frequency ω as an example:

$$c(t) = \cos \omega t. \tag{13}$$

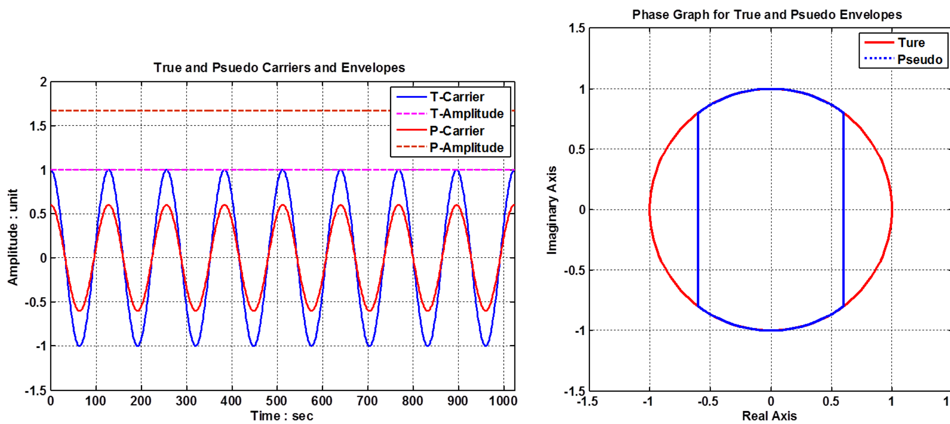
If we adopt the constant value 0.6 for $b(t)$ in Eq. (3), then Eq. (13) becomes

$$c(t) = \frac{1}{0.6}(0.6 \cos \omega t), \tag{14}$$

The values of Eqs. (13) and (14) are given in Fig. 1(a). Obviously, the amplitude of Eq. (14) is larger than 1; thus, the amplitude and the carrier is separated and no longer retains the envelope–carrier relationship. If we define the imaginary part of the pseudo-carrier through the direct quadrature method, we would obtain the phase function shown in Fig. 1(b). There, the real part is limited to ± 0.6, beyond which there is no possible value. On the phase plane, the phase function ceases to be a unit circle, but retain only part of the available values. Therefore, the only true amplitude-carrier pair is the one given by Eq. (13); any multiplier other than the unit value constant would destroy this unique relationship.

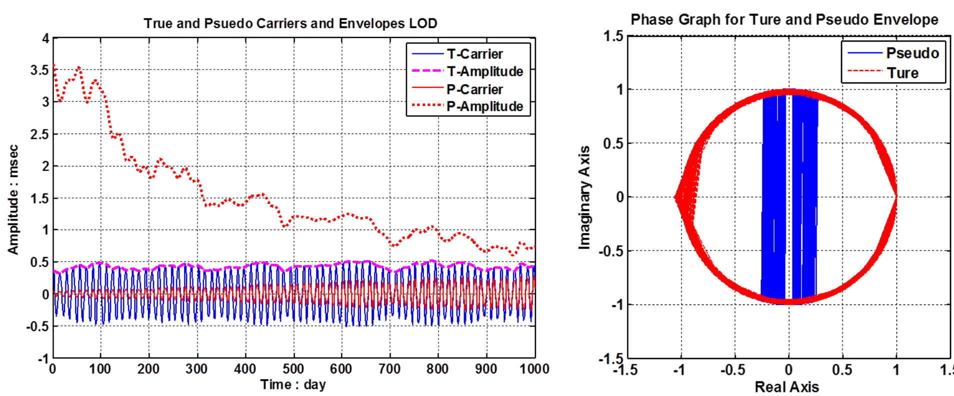
Next, we use a section of the half monthly tidal component from the Length-of-Day data in the second example. This time, we adopt

$$b(t) = 0.1 + 0.0005t, \tag{15}$$



(a) (b)

Fig. 1. (Color online) Numerical results of Example 1. (a) Comparison between the true amplitude-carrier pair and the pseudo amplitude-carrier pair with $b(t) = 0.6$ for the cosine wave $c(t) = \cos \omega t$. (b) Phase function for the pseudo-carrier in (a).



(a) (b)

Fig. 2. (Color online) Numerical results of Example 2. (a) Comparison between the true amplitude-carrier pair and the pseudo amplitude-carrier pair with $b(t) = 0.1 + 0.0005t$ for section of the half monthly tidal component from Length-of-Day data. (b) Phase function for the pesudo-carrier in (a).

Then, the true and pseudo envelope would look like the ones given in Fig. 2(a), where the value of the multiplier $b(t)$ is still less than unity as required over the whole range of 1,000 days. This multiplier again makes the amplitude function to separate from the carrier and cease to maintain the envelope–carrier relationship. Any multiplier with value less than unity would make the real part on the phase plane cease to cover the full ± 1 range. The phase function would also cease to be a unit circle as shown in Fig. 2(b). With these examples, the effect of the multiplier is clearly demonstrated. Thus, the true mathematically meaningful amplitude-carrier pair has to satisfy the envelope–carrier relationship as given by Eq. (1). They could thereby be uniquely defined.

4. The Uniqueness of IF from IMF

The functional form given in Eq. (1) is the ideal case for an IMF obtained through the EMD method using the cubic spline function [Huang *et al.* (1998)], which only gives an approximation. According to Huang *et al.*, the EMD is based on a successive means of the proto-upper and lower envelopes to generate a function that fulfills such two conditions: (1) the number of the extrema and the number of zero-crossings is either equal or differ at most by 1; (2) the mean value of the envelope defined by the local maxima and the envelope defined by the local minima equals to zero.

But as was proved by Wang [Wang *et al.* (2010)] that the exactly symmetric IMF will have constant straight lines for envelopes, the asymptotic state of EMD using infinite number of iterations would indeed produce such an IMF [Wu and Huang (2010)]. Based on this study, the stoppage criterion for EMD, should be preset with the iterative sifting number being at 10 that helps the EMD to have a dyadic decomposition. Using this stoppage criterion, the IMF shown in Fig. 3(a) behaves with only the approximate symmetry within the tolerable range as stipulated in the stoppage criteria stipulated here or any of its variations. Such an approximate envelope would cause difficulty in computing the IF as discussed by Huang *et al.* [2009]. A symmetric envelope for the approximate IMFs could be derived easily through the normalization procedure proposed in Huang *et al.* [2009] as follows: first, take the absolute value of the IMF and find the cubic spline proto-envelope going through all the local maxima as $e_1(t)$ shown in Fig. 3(b); next, normalize the IMF $y_1(t)$ through:

$$y_1(t) = \frac{|x(t)|}{e_1(t)}. \tag{16}$$

Here, the values of $y_1(t)$ should be unity at all the maxima, but they may not the tangential points. Ideally, the values of $y_1(t)$ should be all less than 1. In case that the proto-envelope cuts into the data function $x(t)$, there will be portions of $y_1(t)$ greater than unity as in Fig. 3(c). Then we can divide $y_1(t)$ into wave element functions, and construct a second order proto-upper envelope $e_2(t)$ that would have less chance of the envelope cutting into the data. This procedure could be iterated n-times by finding successive proto-envelope till $y_n(t) \ll 1$ for all t:

$$y_2(t) = \frac{y_1(t)}{e_2(t)},$$

$$\vdots$$

$$y_n(t) = \frac{y_{n-1}(t)}{e_n(t)}. \tag{17}$$

Eventually, we would eliminate all the possibilities of the envelope cutting into the data. In practice, most function would be normalized after two or three trials. In

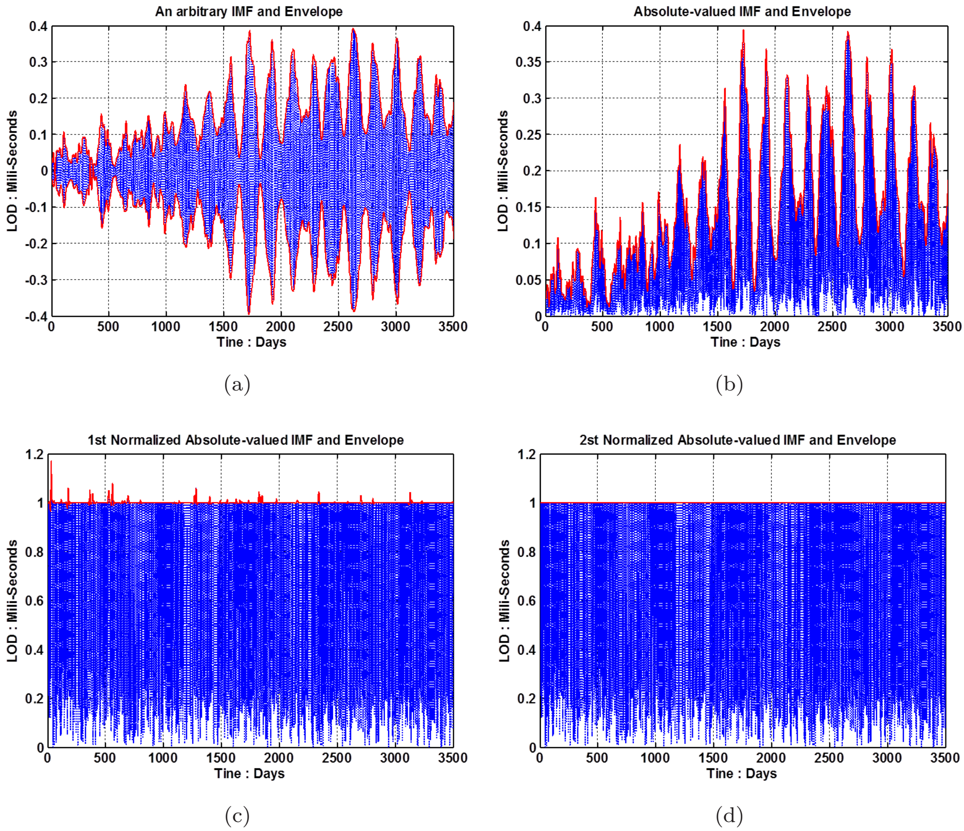
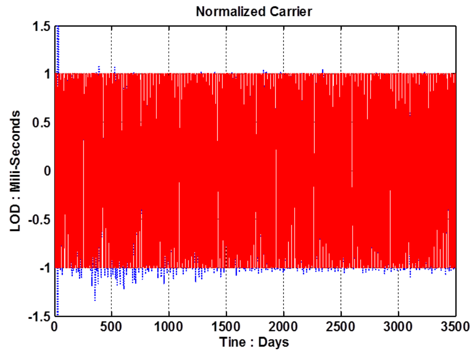


Fig. 3. (Color online) (a) IMF with only approximate symmetry within the tolerable range and one of its arbitrary envelope. (b) Absolute-valued envelope of the IMF in (a). (c) The first normalized absolute-valued IMF in (a) and the corresponding normalized absolute-valued envelope. (d) The second normalized absolute-valued IMF in (a) and the corresponding normalized absolute-valued envelope.

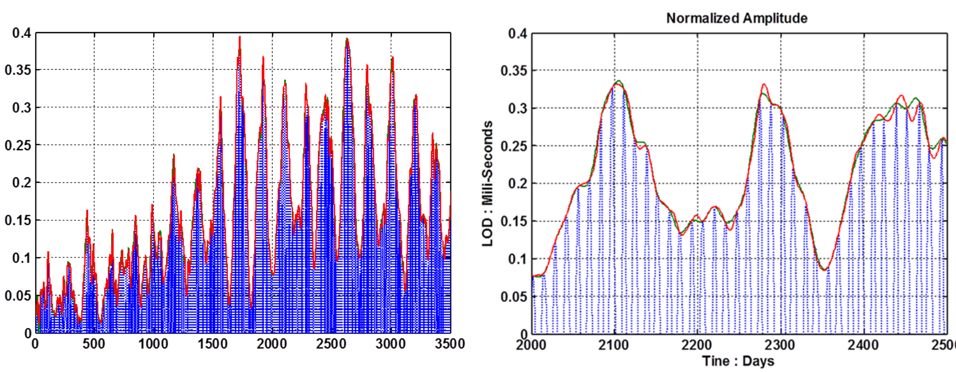
the present case, the second iteration shown in Fig. 3(d), is sufficient. Thus we would obtain the envelope E of the absolute function as

$$E(t) = e_1(t) \cdot e_2(t) \ldots \cdot e_n(t), \tag{18}$$

This envelope function should be used as the amplitude function. Figure 4(a) shows the carrier functions: at end of the first normalizing step, it is not all within ± 1, which would generate in IF computation. But after the second iteration, the carrier function all falls within ± 1. The normalized amplitude function is shown in Fig. 4(b) with more details. These steps and the figures actually demonstrate that the approximate IMF could be normalized and assumed a form that would be expressed as $a(t) \cos \theta(t)$. Mathematically, the normalized IMF satisfies Eq. (1). Such a pair of amplitude and carrier is unique to the data. It is intrinsic to the IMF; therefore, we designate this envelope as the intrinsic amplitude and the carrier as the intrinsic



(a)



(b)

Fig. 4. (Color online) (a) Normalized carrier of the IMF in Fig. 3(a). (b) Left: Amplitude of the IMF in Fig. 3(a) in the first normalizing step; Right: Normalized amplitude of the IMF in Fig. 3(a).

carrier. The intrinsic envelope of an IMF can uniquely define the amplitude with respect to the class $\mathbb{C}^2$ functional used. It is straightforward that the amplitude can relate to the carrier through a complex form:

$$z(t) = x(t) + \mathrm{i}d(t). \tag{19}$$

To ensure the phase function locating on an unit circle in the complex phase plan, the function $d(t)$ can be easily computed by the direct quadrature [Huang *et al.* (2009)]:

$$\frac{d(t)}{a(t)} = \sqrt{1 - \left[\frac{x(t)}{e_1(t) \cdot e_2(t) \ldots \cdot e_n(t)}\right]^2}. \tag{20}$$

Thus

$$\frac{x^2(t) + d^2(t)}{E^2(t)} = 1. \tag{21}$$

In contrast, the Hilbert transform cannot guarantee the above requirement, i.e.

$$\frac{c^2(t) + [\mathcal{H}(c(t))]^2}{a^2(t)} \neq 1, \tag{22}$$

unless the function $c(t)$ satisfies the theorems proposed by Bedrosian [1963] and Nuttall [1966]. In general, the complex valued data through Hilbert transform would not satisfy the unit circle requirement. Therefore, the Hilbert transform should not be used as a genuine method to compute the IF, for both the amplitude and carrier may be distorted.

5. Discussion

It should be pointed out that what we have established here is the uniqueness of the IF in the narrow sense: it is true only with respect to a given $\mathbb{C}^2$ family of plane curves in defining the envelope, subject to the variation given in Eqs. (3) and (4). Therefore, this proof has some limitations. In the above discussion, the envelope is defined through cubic spline function. Of course, there are many different kinds of $\mathbb{C}^2$ family of plane curves. Other type of spline such as the Hermitian spline could be used, of course. But, any special spline adopted would call for additional condition other than the basic continuity assumption adopted in the natural spline and would, therefore, be less adaptive. Furthermore, any alternative $\mathbb{C}^2$-class curve would only change the final result slightly in quantitative but not in essence.

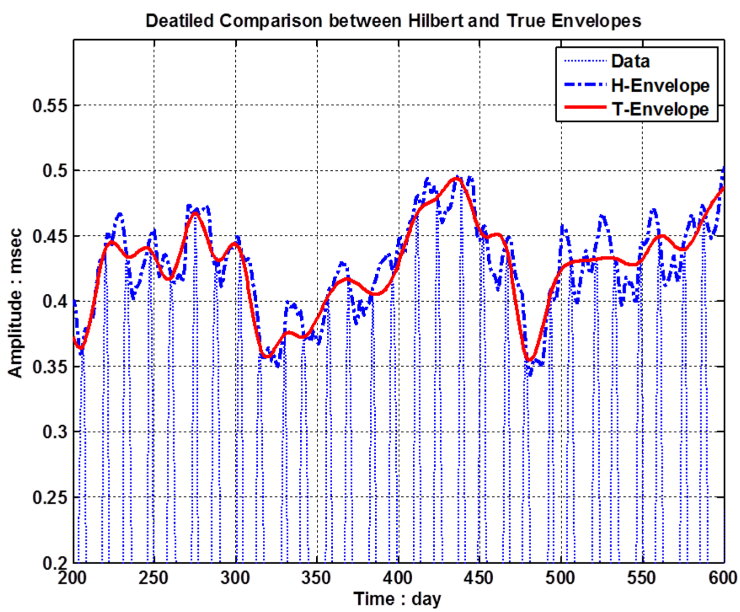


Fig. 5. (Color online) Detailed comparison between the Hilbert and true envelopes.

It should also be pointed out that though Hilbert Transform seems to give an envelope to any mono-component function, it is strongly advised to eschew it here. In addition to the limitation on the IF pointed by Bedrosian and Nuttall, Hilbert Transform would give a very rugged envelope when the signal is not strictly linear as discussed by Huang *et al.* [Huang *et al.* (2006)], or as shown in Fig. 5.

By the way, there are other objections to the IF due to the uncertainty principle [Gröchenig (2001)], which stipulates that the time and frequency in a spectrum could not be made arbitrarily precise, and then how can we talk about the IF? This fundamental statement for the Fourier transform pair is certainly true. As is known, the uncertainty principle in physics has been well grounded, while in data analysis, it is actually the consequence of integral transform contained in the Fourier or the more general wavelet analyses. In contrast, the method proposed by Huang *et al.* [Huang *et al.* (1998, 2006)] eschews the integral transform and thus resolves this issue automatically.

The major contribution of this short note is to establish that the expression given in Eqs. (1) could be made unique through Eqs. (9)–(11) or Eq. (17) through Eq. (19), additionally, a meaningful IF could also be computed. Simple methods are proposed to test whether an expression is or is not the true and intrinsic amplitude-carrier pair. Accoding to Huang *et al.* [2006], the normalization may shift the location of the extrema if the amplitude fluctuation is large. Unfortunately, no criteria could be established here for the existence of such condition. The slight shifting of the extrema is a price we have to pay here.

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Appendix A. Construction of a $\mathbb{C}^2$-Envelope to any Oscillatory Function

Given a family of curves in the plane, an envelope is a curve tangent to each member of the family at some point. Rigorous as the definition is, there is no established general method to find such a curve, unless the given family of functional form is given. Here, we present a constructive method for a $\mathbb{C}^2$-envelope applicable to the scattered numerically valued data. We start from the unique data plotted as function $x(t)$ that may be any function with multiple interposed local extrema (with alternative maxima and minima) as follows. First, divide the oscillatory function into sections by consecutive minima (or maxima), then designated each section as a wave element function as shown in Fig. A.1. These wave element functions constitute a

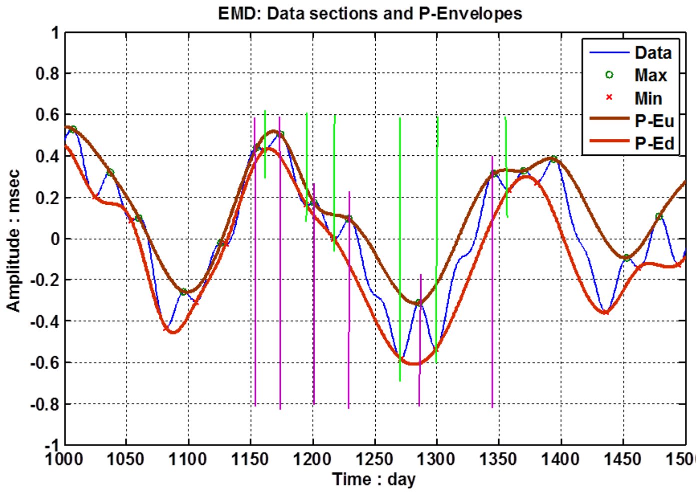

Fig. A.1. (Color online) Data sections and pseudo-envelope.

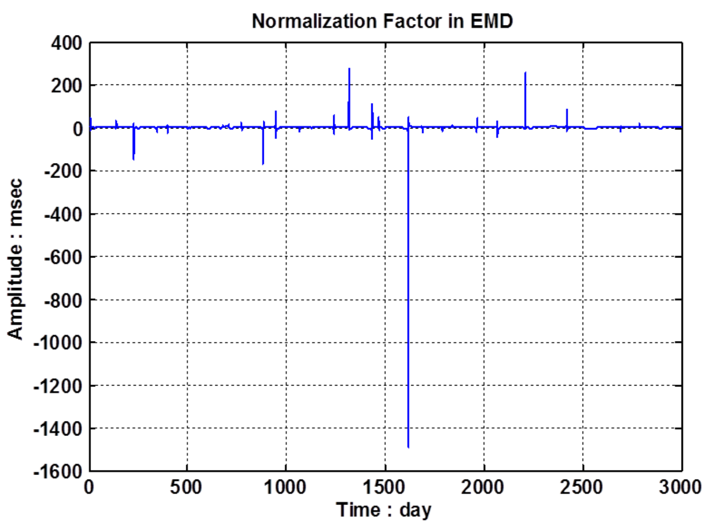

Fig. A.2. (Color online) Normalization factor in EMD.

family of curves in the plane. Now, let us define a upper (lower) proto-envelope $e_1(t)$ as a class $\mathbb{C}^2$-curve (natural cubic spline, for example) passing through all the local maxima (or minima). By construction, the upper (lower) proto-envelope should be close to the upper (lower) bound of the function, which however cannot be guaranteed for the upper (lower) proto-envelope might not be tangential to all wave element functions. They can actually cut through the data at some portion of the curve.

To eliminate those cutting through portion, the straightforward application of the normalization process to construct a true envelope would run into difficulties for two reasons: first, the initial step of defining the first proto-envelope $e_1(t)$ could have both positive and negative values; next, as the first proto-envelope $e_1(t)$ contains values close to zero, the normalization step will produce many large numbers as shown in Fig. A.2.

To overcome this difficulty, we suggest a simple shift of the reference coordinate by a positive constant value m, so that all the maxima are positive and away from the zero axis. Thus, we obtain an expression that is slightly different from Eq. (A.1):

$$y_1(t) = \frac{x(t)}{e_1(t)}. \tag{A.1}$$

As this simple shift would not alter the geometry of the envelope, we can normalize the proto-envelope in this shifted coordinate. At the end, the true envelope could be determined by shifting the result back the identical amount m to the original coordinate. The effect of this shift is employed only to reduce the fluctuation of $y_1(t)$. When $e_1(t) - x(t) > 0$, $y_1(t)$ would be less than unity; therefore, the increase of m value will increase the value of $y_1(t)$ to push it closer towards unity. When $e_1(t) - x(t) < 0$, $y_1(t)$ would be greater than unity; therefore, the increase of m value will decrease the value of $y_1(t)$ again to push it towards unity. The net effect is to reduce the fluctuation of $y_1(t)$, and make the normalization process converges faster. Whatever the amount of shifting in the coordinate, we can always shift back. The final resulting envelopes should be identical after the normalization.

The envelopes for the EMD are based on proto-envelopes. It could be defined rigorously based on the method outlined in Appendix A. With the present definition of envelope, we can revisit the EMD as proposed by Huang *et al.* [Huang *et al.* (1998)]. The class $\mathbb{C}^2$-function used is the natural cubic spline and the envelope is simply the proto-envelope. There is no normalization steps. Even the definition for the IMF is given in terms of the proto-envelopes. We can now tighten the procedures of EMD and the definition of IMFs by using the true envelopes. Considering the time consuming computation and the close approximation we obtained from the proto-envelope, we still recommend that the proto-envelope approach as proposed in Huang *et al.* [1998].

With the definition of true envelope, however, would enable us to simplify the definition of IMF. To do this, we first define a mono-component function as any function having the same numbers of extrema and zero crossings or differ at most by one, which is precisely the first condition given in Huang *et al.* [1998]. Then, an IMF can be simply defined as a mono-component function with symmetric upper and lower envelopes.

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